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EE109: Convex Optimization

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Final Project – Problem Formulation

**Volume Forecasting for Multi-Period Trading via Convex Optimization**

**Background**

In 1952 Harry Markowitz published his seminal paper “Portfolio selection” and gave birth to the field of portfolio theory. In the preceding years many developments have been built on his work, yielding increasing levels of insight into financial market functionality and financial decision making. The fundamental question answered by his paper is: How should an investor allocate funds among a possible set of asset choices?

To address this question Markowitz suggested two novel approaches:

1. A method to quantify the risk and return of an asset using statistics (variance, expected value)
2. Investors should consider risk and return together and determine the allocation of funds based off of a risk-return profile of their portfolio.

Before the introduction of his theory, investors used risk and return in a disjointed fashion, rather than treating them as a connected trade-off between one measurement and the other. This approach has provided great value to investors. By quantifying and treating risk and return as a single metric, investors could then evaluate a portfolio’s return based off of both the individual potential returns of an asset and the relationship of assets returns to each other, rather than viewing the returns of a portfolio as the weighted average of the individual returns.

A simple example of the pre and post portfolio theory approach:

Take a portfolio composed of oil, transportation, and industrial manufacturing securities. The pre portfolio theory approach would be analyzing the individual returns of the assets. Our qualitative analysis of the companies indicates that all three of the securities will increase in value, and when we quantify and project the financial growth, we find the companies are projected to grow by the same amount. In this case we would want to purchase only one of the securities to minimize the cost of purchasing the security (a broker is required to link a buyer and seller in the market – known as market makers, and they charge a fee per order for their service).

Using the methods of portfolio theory an investor would likely find a connection between the three companies. In this case, if the price of oil increases, the cost of transportation will increase and decrease the value or return of our transportation security. The rise of oil price will likely increase the value of our oil drilling component manufacturer and the share price will rise together. Our analysis of the individual securities implied that all three will grow, but the connection between the industries implies that their growths are linked to each other and our financial analysis is likely risky for each of them. In order to reduce the risk of the portfolio we would want to distribute roughly half our funds to the transportation company and half between the oil and oil component manufacturer. In this case we hedge our projections against each other and reduce the risk of an overall loss, while maximizing the return.

What portfolio theory allows us to do is extrapolate this approach to a wide variety of assets in a highly complicated and inter-connected system. This is especially useful where the connections between assets may not be as readily apparent, e.g. adding corn to your portfolio may reduce the risk associated with investing in an aerospace security because there may be an illogical correlation between the two. Applying this method across a portfolio allows investors to generate higher returns at a lower risk.

Markowitz formulated portfolio theory as an optimization problem as mean variance optimization (MVO). The MVO has an infinite number of potential portfolios to select from to generate a desired return, and then from there the risk is minimized. The formulation of this problem is in the appendix. To apply portfolio theory in practice investors will utilize quantitative trading algorithms. These algorithms have become increasingly popular over the past decade as the cost of computation has decreased. One of the problems to develop from portfolio MVO is that the model does not include the cost of transaction for each of the stocks.

To address this a series of new trading algorithms have been developed. Most of the work done in this field has been on approaching the problem from a single period point of view, referred to as Single Period Optimization (SPO). These SPO techniques give investors guidance as to which and how much of the funds should be distributed to a selection of assets, effectively performing the same analysis as MVO but adding additional information on the transaction and hold costs of assets. A limitation in the technique is it does not take into account the future returns of the portfolio. To address this a new field of Multi-Period Optimization (MPO) has been created to address this. It answers the question for investors of when each of the assets should be purchased and has become more viable as computation has become less costly.

**Assumptions – Mean Variance Optimization**

Markowitz’s theory of portfolio selection is based on several assumptions:

1. Risk of a portfolio is based on the variability of returns from the said portfolio
2. An investor is risk averse
3. An investor is rational
4. An investor prefers to increase consumption
5. An investor’s utility function is concave and increasing, due to his risk aversion and consumption preference.
6. A portfolio is selected for a single period
7. An investor either minimizes risk for a specified return or maximizes returns for a specified risk

For the single period trading algorithms, all of the same assumptions are made as for mean variance optimization, however, the investor preference to increased consumption is removed. This is done to allow investors to analyze trading and holding costs for assets, making long-short mixed portfolios an option (because MVO often leads to portfolios where large shorts are taken to increase long positions). In the multi-period framework, the same assumptions as the single period hold true less the single period constraint.

For the single period case, there is an assumption that the returns from period to period are independent and drawn from the same distribution, and that the historical data from period to period can give insight into the distribution. These assumptions are then lifted in the multi-period model.

**Problem Formulation**

*Mean Variance Optimization*

We consider a portfolio of n securities, S1, 2, …, n, and forecasted returns r­1, 2, …, n, represented in vector form as r = [r1, r2, …, rn]T . We have a total amount of funds to invest F, and distribute it across the portfolio of securities, S1, 2, …, n, with h1, 2, …, n representing the dollar amount of each asset holding, and a weight vector, w = [w1, w2, …, wn ]T , where wi represents the proportion of the total funds allocated to asset i, wi = hi/F, and a trade vector x, where xi represents the dollar value trade of asset i and z is the normalized trade vector, z = x/(1Th). This makes the overall return of the portfolio rp(w) = wTr. We can define σI as the standard deviation for asset i, ρ­ij as the correlation coefficient between returns of asset i and asset j. We can then define an n x n covariance matrix:

Where σii = σi2 and σij = σji = ρijσ­iσj (for i ≠ j). We can compute the variance of the portfolio as V(w) = wTΣw. We can also represent the expected value of the portfolio as µ = [µ1, µ2, …, µn]T, where µi is the expected return of asset i.

There’s three ways the problem can be formulated. If the investor is trying to maximize return subject to a maximum risk, the problem becomes:

To minimize risk subject to a minimum desired return:

These two can be combined into a single unconstrained minimization problem where the investor choses a risk aversion parameter λ:

*Single/Multi-Period Trading*

We introduce transaction and holding costs to the model as and . These will be determined based off the type of asset being analyzed and desired outcomes by the portfolio manager, but fit the general form:

Where a is one half the bid-ask spread price expressed as a fraction of the unit price, b is a positive constant, V is the market volume traded for the asset in a time period, sigma is corresponding volatility for recent periods, and c is used to create asymmetry in the cost function (for c=0, the trading cost of buying and selling is the same). In the holding cost function, stT represents the fee charged for shorting an asset, ht+ represents the holdings of a portfolio after executing trades for the current period (previous holdings plus current trades), and .

We also introduce a self-financing constraint to the portfolio:

And can normalize the constraint as:

Then formulate the optimization problem as:

Subject to the self-financing constraint. In this equation ψ is a risk function. The objective function of this problem is concave (return is concave, cost functions are convex, and risk function is convex). The self-financing constraint is non-convex; however, the constraint can be loosened to the inequality:

In maximizing the returns all of the cash will be utilized and therefore the self-financing constraint will be equal at the optimal solution of the optimization problem.

In the multi-period case, we want to do our analysis over a planning horizon that extends H periods into the future, t, t+1, …, t+H-1. Since we are assuming, we are working with a large portfolio, we can further relax the self-financing constraint by assuming the cost functions are negligible relative to the value of the trades. This relaxed self-financing constraint becomes 1Tz = 0. Our multi period optimization becomes:

subject to:

Since the multi-period trading function is a positively weighted combination of single period optimization functions, then it is also known to be convex.

**Key References:**

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